Exam Percolation Theory

1 November 2024, 8:30-10:30

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (25 pts).
State and prove the (correctness of the) Margulis-Russo formula.

(*Hint*: just to jog your memory, this formula relates the derivative wrt. p of the probability of a certain kind of event to "influences".)

Exercise 2 (a:9,b:8,c:8 pts).
In a country far away, an election is about to be held between two candidates. In the setting of the current exercise, the candidates are represented by -1, respectively $+1$. There are n voters, whose votes can be accurately modelled by independent flips of a fair coin. Let the random vector $X = (X_1, \ldots, X_n) \in \{\pm 1\}^n$ represent their votes. (So X_i are i.i.d. with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$.) The outcome of the election is $f(X)$ where f is some (possibly very unusual) function $f : \{\pm 1\}^n \to \{\pm 1\}$.

Let the random variable

$$W := |\{i : X_i = f(X)\}|,$$

equal the number of voters that have voted for the winner. Recall that $\hat{f}(\{i\})$ denotes the coefficient of x_i in the Fourier-Walsh decomposition of f.

a) Show that $\mathbb{E}\Big(f(X)(X_1 + \cdots + X_n)\Big) = \sum_{i=1}^n \hat{f}(\{i\})$.

(*Hint*: Make sure to state clearly any results from the lecture notes you are using.)

b) Show that $\mathbb{E}W = \frac{n}{2} + \frac{1}{2}\sum_{i=1}^n \hat{f}(\{i\})$.

(*Hint*: Express $\mathbb{E}\Big(f(X)(X_1 + \cdots + X_n)\Big)$ in terms of of W and n and compare with part **a)**.)

In 1762, the French philosopher Rousseau suggested that the best election rule is the one that maximizes the number of voters that agree with the outcome. The majority function $g : \{\pm 1\}^n \to \{\pm 1\}$ is given by:

$$g(x_1, \ldots, x_n) := \begin{cases} +1 & \text{if } x_1 + \cdots + x_n > 0, \\ -1 & \text{otherwise.} \end{cases}.$$

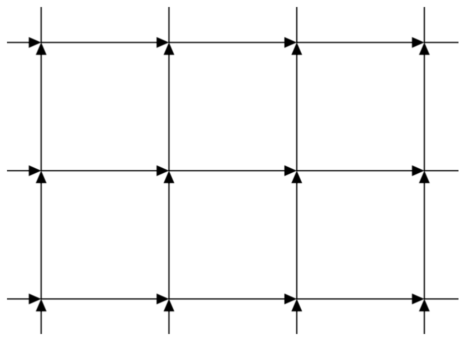
(Notice that g corresponds to what most of us would consider the natural way to decide the winner of the election, at least when there is no tie.)

c) Show that the majority function g maximizes $\mathbb{E}W$ among all "election rules" $f : \{\pm 1\}^n \to \{\pm 1\}$.

(*Hint*: How does $f(x) \cdot (x_1 + \cdots + x_n)$ relate to $|x_1 + \cdots + x_n|$?)

(See next page)

Exercise 3 (a:5, b:5, c:5, d:5, e:5 pts)
In this exercise we consider *oriented percolation* on $\mathbb{Z}^2$. This refers to the situation where each edge has a direction, and the liquid can only pass in that direction. The horizontal edges are directed to the right, and the vertical edges are directed upwards. As in the standard percolation model, every edge is open with probability p, independently of all other edges.



Naturally, we say that "(oriented) percolation occurs" if there is some open, infinite, directed path.

Let us set

$$p_c(\mathbb{Z}^2, \text{oriented}) := \inf\{p \in [0,1] : \mathbb{P}_p(\text{oriented percolation occurs}) > 0\}.$$

a) Explain why $p_c(\mathbb{Z}^2, \text{oriented}) \geq 1/2$.

(*Hint*: one or two well-chosen sentences should suffice.)

A natural question that presents itself, is whether oriented percolation is non-trivial. That is, is $p_c(\mathbb{Z}^2, \text{oriented}) < 1$?
It turns out the answer is "yes", as you will hopefully be able to show in the next part.

b) Show that $p_c(\mathbb{Z}^2, \text{oriented}) < 1$.

(*Hint*: You could try to adapt the argument of Broadbent and Hammersley that $p_c(\mathbb{Z}^d) \leq 2/3$ for $d \geq 2$. You may use without proof any statements about contours etc. from the lecture notes, provided you state them clearly and correctly. If $\mathcal{C}$ denotes the set of all nodes reachable by open, directed paths from the origin and the contour of $\mathcal{C}$ has length n, then how many of the directed edges the contour crosses must be closed?)

In the light of part **a)**, another natural question is whether $p_c(\mathbb{Z}^2, \text{oriented}) > 1/2$ or not. In order to answer that question, we define for each $n \in \mathbb{N}$:

$$X_n := \left| \left\{ (x,y) \in \mathbb{Z}^2 : \begin{array}{l} x + y = n \text{ and there is an open, directed} \\ \text{path starting at } (0,0) \text{ and ending in } (x,y) \end{array} \right\} \right|.$$

c) Show that $\mathbb{E}X_2 = 4p^2 - p^4$.

d) Show that $\mathbb{E}X_{2n} \leq \left(4p^2 - p^4\right)^n$, for every $n \in \mathbb{N}$.

e) Show that

$$p_c(\mathbb{Z}^2, \text{oriented}) \geq \sqrt{2 - \sqrt{3}} \approx .5176$$

(*Hint*: Use the previous part. Make sure your reasoning is correct and very clear. You are not asked to justify the stated numerical approximation.)

(See next page)

Exercise 4 (a:5, b:5, c:5 pts)
In this exercise, we consider the following *power-law long-range percolation model* on $\mathbb{Z}^d$, with $d \geq 2$. The base graph is the complete graph on vertex set $\mathbb{Z}^d$. (I.e. every pair of nodes is connected by an edge.) The events of edges being open are independent, but the probabilities of being open are not the same for every edge. Specifically, for $x, y \in \mathbb{Z}^d$ with $x \neq y$ the edge between x, y is open with probability

$$p(x, y) := \alpha \cdot \|x - y\|^{-\beta},$$

where $0 < \alpha < 1$ and $\beta > 0$ are parameters of the model. As usual we say percolation occurs if there is some infinite open path. For any $\beta > 0$, we define $\alpha_c = \alpha_c(\beta)$ by:

$$\alpha_c := \inf\{0 < \alpha < 1 : \mathbb{P}_{\alpha,\beta}(\text{percolation}) > 0\}.$$

a) Show that

$$s(\beta) := \sum_{z \in \mathbb{Z}^d \setminus \{\underline{0}\}} \|z\|^{-\beta} = \begin{cases} < \infty & \text{if } \beta > d, \\ \infty & \text{if } 0 < \beta \leq d. \end{cases}$$

(*Hint*: You may use without proof that there exist $c_1, c_2 > 0$ such that $c_1 n^{d-1} \leq |\partial \Lambda_n| \leq c_2 n^{d-1}$ for all n. What can you say about $\|z\|$ for $z \in \partial \Lambda_n$?)

b) Show that $0 < \alpha_c < 1$ if $\beta > d$.

c) Show that $\alpha_c = 0$ if $0 < \beta \leq d$.

(The end)